Numerical Solution of Twelfth Order Boundary Value Problems Using Homotopy Analysis Method

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Abstract
In this paper, we demonstrate homotopy analysis method (HAM) which is a powerful and easy-to-use technique to solve linear and non-linear boundary value problems. HAM contains an auxiliary parameter $\hbar$ which provides a convenient way to control the convergence region of the series solutions. Two examples are presented to illustrate the effectiveness of HAM for solving twelfth order boundary value problems. By comparing the obtained results with the existing results, it is concluded that HAM is a powerful tool for solving twelfth order boundary value problems arising in various fields of engineering science.

Introduction
Many of the mathematical models which arise in the study of hydrodynamic, hydro magnetic stability, fluid dynamics, engineering and applied physics are expressed in terms of boundary value problems, which are ordinary differential equations with boundary conditions. However, it is usually difficult to obtain the closed-form solutions of such boundary value problems, especially for non-linear boundary value problems. In most cases, only approximate solutions (either numerical solutions or analytical solutions) can be expected. Some numerical methods such as finite difference method, finite element method, etc., have been developed for obtaining the approximate solutions of the boundary value problems.

Perturbation methods [1] for solving non-linear problems are based on the existence of small/large parameters, the so-called perturbation quantity, whereas, non-perturbation methods such as differential transformation method and Adomian’s decomposition method are independent of these small parameters, but cannot provide us with a simple way to adjust or control the convergence region and rate of given approximate series.

Liao [2] proposed a powerful analytic method for non-linear problems, namely the homotopy analysis method [3-4] which is a general analytic approach to get the series solutions of various types of non-linear equations, including ordinary differential equations, partial differential equations, differential-integral equations, differential-difference equations and coupled equations. Unlike the perturbation and non-perturbation techniques, HAM itself provides us with a convenient way to control and adjust the convergence region and rate of approximation series. The method is valid even if a given non-linear problem does not contain any parameter. It can be employed to efficiently approximate a non-linear problem by choosing different sets of base functions. These advantages make the method to be a powerful and flexible tool in mathematics and engineering, which can be readily distinguished from existing numerical and analytical methods. A systematic description of the method and its applications are found in [5].

Twelfth order differential equations have several important applications in engineering. Chandrasekhar [6] showed that when an infinite horizontal layer of fluid is put into rotation, subject to heat from below and a uniform magnetic field across the fluid in the same direction as gravity, instability will occur. It was then shown that if instability sets in as over stability, the problem is governed by a twelfth order boundary value problem. Such problems arise in geophysics when studying core fluid adjacent to the core-mantle boundary. Several researchers developed numerical techniques for solving twelfth order differential equations. For example, Twizell et al. [7] developed finite difference methods for solving high-order eigen value problems arising in thermal instability, Islam et al.[8] used differential transform method.

In this paper, we apply HAM to obtain the approximate solution of twelfth order boundary value problems of the type

$$y^{(12)}(x) + f(x)y(x) = g(x), \quad x \in [a, b] \quad (1)$$

with boundary conditions

$$y(a) = a_0, \quad y(b) = a_1, \quad y^{(1)}(a) = y_0, y^{(1)}(b) = y_1, \quad y^{(2)}(a) = \delta_0, y^{(2)}(b) = \delta_1, \quad y^{(3)}(a) = v_0, y^{(3)}(b) = v_1, \quad y^{(4)}(a) = \beta_0, y^{(4)}(b) = \beta_1, \quad y^{(5)}(a) = \omega_0, y^{(5)}(b) = \omega_1 \quad (2)$$
where \(a_i, \gamma_i, \delta_i, v_i, \beta_i\), and \(d_{ai}\), \(i = 0, 1\) are finite real constants and the functions \(f(x)\) and \(g(x)\) are continuous on \([a, b]\). The rest of the paper is organized as follows. Basic idea of HAM is presented in Section 2. Section 3 presents numerical examples to assess the efficiency of HAM. Finally, Section 4 concludes the paper.

2. Basic Idea of HAM:
To have a basic idea of HAM, let us consider the following differential equation

\[ N[u(x)] = 0, \quad (3) \]

where \(N\) is a non-linear operator, \(x\) denotes independent variable, \(u(x)\) is an unknown function. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. By means of generalizing the traditional homotopy method, Liao [9] constructed the following zeroth order deformation equation,

\[ (1 - p)L(\phi(x; p) - u_0(x)) = p\hbar H(x)N(\phi(x; p)), \quad (4) \]

where \(p \in [0, 1]\) is the embedding parameter, \(\hbar \neq 0\) is a nonzero auxiliary parameter, \(H(x) \neq 0\) is an auxiliary function, \(L\) is an auxiliary linear operator, \(u_0(x)\) is an initial guess of \(u(x)\), \(\phi(x; p)\) is an unknown function of the independent variables \(x\) and \(p\).

It is important to note that one has great freedom to choose the above mentioned auxiliary functions and parameters in HAM. When \(p = 0\) and \(\hbar = 1\), \(\phi(x; p)\) becomes

\[ \phi(x; 0) = u_0(x), \quad \phi(x; 1) = u(x), \quad (5) \]

respectively. Thus, as \(p\) increases from 0 to 1, the solution \(\phi(x; p)\) varies from the initial guess \(u_0(x)\) to the solution \(u(x)\). Expanding \(\phi(x; p)\) in Taylor series with respect to \(p\), we have

\[ \phi(x; p) = u_0(x) + \sum_{m=1}^{\infty} u_m(x)p^m, \quad (6) \]

where \(u_m(x) = \frac{1}{m!} \left. \frac{\partial^m \phi(x;p)}{\partial p^m} \right|_{p=0} \quad (7) \)

If the auxiliary linear operator, the initial guess, the auxiliary parameter and the auxiliary function are properly chosen, series (6) converges at \(p = 1\) and we have

\[ u(x) = u_0(x) + \sum_{m=1}^{\infty} u_m(x), \quad (8) \]

which must be one of the solutions of the original non-linear equation as proved by Liao [9]. Define the vector \(\overline{u}_n\) as \(\overline{u}_n = \{u_0, u_1, \ldots, u_n\}\). Differentiating equation (4) \(m\) times with respect to the embedding parameter \(p\), setting \(p = 0\) and finally dividing with \(m!\), we obtain the \(m^{th}\) order deformation equation as

\[ L(u_m - \chi_m u_{m-1}) = \hbar H(x)R_m(\overline{u}_{m-1}), \quad (9) \]

Where

\[ R_m(\overline{u}_{m-1}) = \frac{1}{(m-1)!} \left. \frac{\partial^{m-1} \phi(x; p)}{\partial p^{m-1}} \right|_{p=0}, \quad (10) \]

and

\[ \chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases} \quad (11) \]

Thus, we can obtain \(u_0, u_1, \ldots, u_n\) by solving the linear higher order differential equation (9) one after the other. The \(m^{th}\) order approximation of \(u(x)\) is given by \(u(x) = \sum_{m=0}^{\infty} u_m(x)\).

3. Numerical Examples:
In order to illustrate the ability of HAM in solving twelfth order boundary value problems, we consider the following examples. The obtained HAM results are compared with the existing results in literature and are seen to be in good agreement.

**Example 1:** For \(x \in [-1, 1]\), consider the linear twelfth order boundary value problem [10]

\[ y^{(12)}(x) = -12(2x \cos(x) + 11 \sin(x)) + y(x), \quad (12) \]

with boundary conditions

\[
\begin{align*}
 y(-1) &= y(1) = 0, \\
 y^{(1)}(-1) &= y^{(1)}(1) = 2 \sin(1), \\
 y^{(2)}(-1) &= -y^{(2)}(1) = -4 \cos(1) - 2 \sin(1), \\
 y^{(3)}(-1) &= y^{(3)}(1) = 6 \cos(1) - 6 \sin(1), \\
 y^{(4)}(-1) &= -y^{(4)}(1) = 8 \cos(1) + 12 \sin(1), \\
 y^{(5)}(-1) &= y^{(5)}(1) = -20 \cos(1) + 10 \sin(1).
\end{align*}
\]

Exact solution of the above differential equation is

\[ y(x) = (x^2 - 1) \sin(x). \quad (14) \]

For the zeroth order deformation equation (4) the auxiliary linear operator is given by

\[ L(\phi(x; p)) = y^{(12)}(x). \quad (15) \]

We also choose \(H(x) = 1\), for simplicity and the non-linear operator \(N\) as

\[ N(\phi(x; p)) = \phi^{(12)}(x) - 12(2x \cos(x) + 11 \sin(x)) + \phi(x). \quad (16) \]

In view of the boundary conditions (13), the initial guess to (12) is obtained as
Following the explanation given in the previous section, the \( m^{th} \) order deformation equation becomes

\[
L(y_m - x_m y_{m-1}) = h R_m(y_{m-1}), \tag{18}
\]

with

\[
\begin{align*}
\gamma^{(-1)} &= y^{(1)}(x) = y^{(0)}(x) = y^{(0)}(x) = y^{(0)}(x) = 0, \\
y^{(1)} &= y^{(0)}(x) = y^{(0)}(x) = y^{(0)}(x) = y^{(0)}(x) = 0. \tag{19}
\end{align*}
\]

Where

\[
R_m(y_{m-1}) = y^{(12)}_{m-1}(x) + (1 - X_m)(12(2x \cos(x) + 11 \sin(x))) + y_{m-1}(x). \tag{20}
\]

Now the approximate solution of (12) can be obtained as

\[
y(x) = -7.64071 \times 10^{-38} - x + 1.29304 \times 10^{-37} x^2 + 1.16667 x^3 - 2.04144 \times 10^{-37} x^4 - 0.175 x^5 + 1.68452 \\
\quad \times 10^{-37} x^6 + 0.00853126 x^7 - 1.19885 \times 10^{-37} x^8 - 0.000200803 x^9 + 1.55534 \times 10^{-38} x^{10} + 2.63285 \times 10^{-5} x^{11}. \tag{23}
\]

And

\[
x_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases} \tag{21}
\]

Now the solution of \( m^{th} \) order deformation equation, for \( m \geq 1 \), becomes

\[
y_m = x_m y_{m-1} + L^{-1}(h \ast R_m(y_{m-1})). \tag{22}
\]

Using symbolic computation software MATLAB, we can recursively obtain \( y_0, y_1, y_2, \ldots \).

Note that the solution series (22) contains the auxiliary parameter, which can be chosen properly by plotting \( h \) - curves to ensure the convergence of the series solution. In Fig. 1, the \( h \) - curve is plotted for 5th order of approximations. It is clear from the figure that \( h = -1 \).
Solution obtained through HAM is compared with exact and HPM (for more details on HPM, see [10]) solutions and are presented in the following table.

Table 1: Comparative Analysis

<table>
<thead>
<tr>
<th>x</th>
<th>Exact Solution</th>
<th>HPM[10]</th>
<th>HAM</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1.0</td>
<td>0.00000000000</td>
<td>-1.60000E-009</td>
<td>0.0000000000</td>
</tr>
<tr>
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<td>0.2582481927</td>
<td>0.2582481925</td>
<td>0.2582481927</td>
</tr>
<tr>
<td>-0.6</td>
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<td>0.3613711820</td>
<td>0.3613711829</td>
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<tr>
<td>-0.4</td>
<td>0.327114075</td>
<td>0.327114041</td>
<td>0.327114073</td>
</tr>
<tr>
<td>-0.2</td>
<td>0.1907225575</td>
<td>0.1907225537</td>
<td>0.1907225573</td>
</tr>
</tbody>
</table>

The above obtained results are plotted in Fig. 2 and from the figure, one can see that the results obtained from HAM are in good agreement with the results of HPM and exact solution.

**Fig. 2** comparison between exact and approximate solutions.

**Example 2:** [To demonstrate HAM for non-linear equations, we considering the following example.]

For \( x \in [0,1] \), consider the non-linear twelfth order boundary value problem [11]

\[
y^{(12)}(x) = (12 + x + x^2e^x)e^x - y^2(x),
\]

with boundary conditions

\[
y(0) = 0, y'(0) = 0, y^{(12)}(0) = 1, y^{(i)}(1) = 2i, \quad i = 1, 2, 3, 4.
\]

The exact solution of the above differential equation is

\[
y(x) = xe^x. \quad (26)
\]

For the zeroth order deformation equation (4) the auxiliary linear operator is given by

\[
L(\phi(x; p)) = y^{(12)}(x), \quad (27)
\]

We also choose \( H(x) = 1 \), for simplicity and the nonlinear operator \( N \) as

\[
N(\phi(x; p)) = \varphi^{(12)}(x; p) - \frac{(1 - \chi_0)(12 + x + x^2e^x)e^x}{\varphi(x; p)}. \quad (28)
\]
In view of boundary conditions (25) the initial guess to (24) is determined as

\[ y_0(x) = 0.00833333 (120x + 120x^2 + 60x^3 + 20x^4 + 5x^5 + 0.999995x^6 + 0.166666x^7 + 0.0237368x^8 + 0.00307044x^9 + 0.000263208x^{10} + 0.0000576459x^{11}), \]

(29)

\[ y(0) = y(1) = y^{(1)}(0) = y^{(2)}(0) = y^{(3)}(0) = y^{(4)}(0) = 0, \]

\[ y^{(2)}(1) = y^{(3)}(1) = y^{(4)}(1) = y^{(5)}(1) = 0, \]

(31)

where

\[ R_m(y_{m-1}) = y^{(12)}_{m-1}(x) - (12 + x + x^2e^x)e^x + \sum_{i=0}^{m-1} y_{m-1}(x) * y_i(x), \]

and

\[ X_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases} \]

(32)

Now the solution of \( m^{th} \) order deformation equation, for \( m \geq 1 \), becomes

\[ y_m = x_m y_{m-1} + L^{-1}(h * R_m(y_{m-1})). \]

(34)

Note that the solution series (34) contains the auxiliary parameter, which can be chosen properly by plotting \( h \)-curves to ensure the solution series convergent. In Fig. 3, the \( h \)-curve is plotted for 5th order of approximations. It is clear from the figure, that the range for admissible values for \( h \) is \(-1.3 \leq h \leq -0.4\). It is evident from our calculations that the series (34) converges in the whole region of \( x \) when \( h = -1 \).

Using symbolic computation software MATLAB, we recursively obtain \( y_0, y_1, y_2, \ldots \)

Following the explanation given in Section 2, the \( m^{th} \) order deformation equation becomes

\[ L(y_m - x_m y_{m-1}) = hR_m(y_{m-1}). \]

(30)

With

\[ y(x) = \begin{cases} -0.00952148 + 0.00952148 e^{2x} - 1.01611x + x e^x - 0.00292969 xe^{2x} - 1.01343x^2 + 0.000244141x^2e^{2x} - 0.507324x^3 - 0.169596x^4 - 0.0425781x^5 - 0.00856116x^6 - 0.00143564x^7 - 0.000205557x^8 - 0.0000266204x^9 - 2.29674 \times 10^{-6}x^{10} - 4.86645 \times 10^{-7}x^{11} - 2.29415 \times 10^{-11}x^{14} - 9.1766 \times 10^{-12}x^{15} - 2.29415 \times 10^{-12}x^{16} - 4.49833 \times 10^{-13}x^{17} - 7.49722 \times 10^{-14}x^{18} - 1.10485 \times 10^{-14}x^{19} + \cdots \end{cases} \]

(35)

The approximate solution of (24) can be obtained by taking only two terms \( y_0, y_1 \) as below

\[ y(x) = -0.00952148 + 0.00952148 e^{2x} - 1.01611x + x e^x - 0.00292969 xe^{2x} - 1.01343x^2 + 0.000244141x^2e^{2x} - 0.507324x^3 - 0.169596x^4 - 0.0425781x^5 - 0.00856116x^6 - 0.00143564x^7 - 0.000205557x^8 - 0.0000266204x^9 - 2.29674 \times 10^{-6}x^{10} - 4.86645 \times 10^{-7}x^{11} - 2.29415 \times 10^{-11}x^{14} - 9.1766 \times 10^{-12}x^{15} - 2.29415 \times 10^{-12}x^{16} - 4.49833 \times 10^{-13}x^{17} - 7.49722 \times 10^{-14}x^{18} - 1.10485 \times 10^{-14}x^{19} + \cdots \]

(35)

Solution obtained through HAM is compared with exact and VIM (for details on VIM, see [11]). solutions and are presented in Table 2. In Fig. 4, one can also see the comparison between obtained results HAM with exact solution and VIM solution.
Table 2: Comparative Analysis.

<table>
<thead>
<tr>
<th>x</th>
<th>Exact Solution</th>
<th>VIM[11]</th>
<th>HAM</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.00000000000</td>
<td>0.0000000000</td>
<td>0.0000000000</td>
</tr>
<tr>
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</tr>
<tr>
<td>0.6</td>
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<td>1.0932712802</td>
<td>1.0932712802</td>
</tr>
<tr>
<td>0.8</td>
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<td>1.0</td>
<td>2.7182818284</td>
<td>2.7182818266</td>
<td>2.7182818284</td>
</tr>
</tbody>
</table>

From Tables 1, 2, Figs.2 and 4, it can be observed that the results obtained using HAM are in good agreement with the results obtained using HPM and VIM.

4. Conclusions:
In this paper, solution of twelfth order boundary value problem has obtained through HAM. The numerical examples considered, revealed that HAM is both accurate and effective for solving twelfth order boundary value problems. It can be concluded that HAM is a highly efficient method for solving high-order boundary value problems arising in various fields of engineering and science. The proposed method can be extended to solve problems that arise in hydrodynamic and hydro magnetic flow of rheological fluids in a contracting or expanding channel (or tube) that accounts heat and mass transfer effects, which are left for future investigation.

References
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