A Coupling Technique for Analytical Solution of Time Fractional Biological Population Model

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ABSTRACT
In this study, homotopy perturbation transform method (HPTM) is used to obtain the approximate analytical solutions of time fractional biological population model. The solution procedure obtained by proposed method indicate that the approach is easy to implement and accurate. Some numerical examples are given in the support of the validity of the method. These results reveal that the proposed method is very effective and easy to use. The comparisons between exact solution and approximate solution are shown through graphs.

Keywords: Homotopy perturbation transform method; Laplace transform; Biological population model; Fractional derivative; Mittag-Leffler function.

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1. Introduction
The idea of fractional-order derivatives initially arose from a letter by Leibnitz to L’ Hospital in 1695. Fractional calculus has gained considerable popularity and importance during the past three decades, mainly due to its applications in numerous fields of science and engineering. One of the main advantages of using fractional-order differential equations in mathematical modelling is their non-local property. It is a well-known fact that the integer-order differential operator is a local operator whereas the fractional-order differential operator is non-local in the sense that the next state of the system depends not only upon its current state but also upon all of its proceeding states. In the last few decades, many authors have made notable contributions to both theory and application of fractional differential equations in areas as diverse as finance [1-2], physics [3-6], control theory [7] and hydrology [8-10].

The degenerate parabolic nonlinear partial differential equations arising in the spatial diffusion of biological populations are given as

\[ u_t = (G(u))_{xx} + (G(u))_{yy} + f(t,x,y,u), \quad t \geq 0, x,y \in \mathbb{R}, \]

(1.1)

with initial condition \( u(x,y,0) = u_0(x,y,0) \), where \( u \) denotes the population density and \( f \) represents the population supply due to birth and death. Our model considered as for example in the population of animals. The movements are made generally either by mature animals driven out by invaders or by young animals just reaching maturity moving out of their parental territory to establish breeding territory of their own. In both cases, it is much more plausible to suppose that they will be directed towards nearby vacant territory. In this model, therefore, movement will take place almost exclusively “down” the population density gradient, and will be much more rapid at high population densities than at low ones. In an attempt to model this situation, they considered a walk through a rectangular grid, in which at each step an animal may either stay at its present location or may move in the direction of lowest population density. In this article, we will do it in a practical case \( G(u) = u^2 \), besides the theory of the spread of biological populations, the case \( G(u) = u^2 \) occurs in a variety of different setting. After this assumption, equation (1.1) will be

\[ u_t = (u^2)_x + (u^2)_y + f(u), \quad t \geq 0, x,y \in \mathbb{R}, \]

(1.2)

with initial condition \( u(x,y,0) = u_0(x,y,0) \).

The homotopy perturbation method was first proposed by J.H. He [11-15]. Considerable research works have been conducted recently in applying the homotopy perturbation method to a class of linear and non-linear equations by many researchers [12-20]. In this article homotopy perturbation transform method has been used. This method was given by Khan and Wu [21] which is coupling of the Laplace transformation, the homotopy perturbation method and He’s polynomials [22-23]. In recent years, many authors have paid attention to studying the solutions of linear and nonlinear partial differential equations by using various methods with combination of the Laplace transform. Among these are the Laplace...
decomposition methods [24-25], homotopy perturbation transform method [26-27].

The main aim of this article is to find the approximate analytical solutions of biological population equation with fractional time derivative of order $\alpha$ ($0 < \alpha \leq 1$). We obtain the solution in the form of a rapidly convergent series with easily computable components by using homotopy perturbation transform method. We consider the more general form of time fractional biological equation by taking $\mu = 1$ as:

$$R_{x(t)} + \mu \frac{\partial^\alpha u(t)}{\partial t^\alpha} = f(t), \quad \alpha > 0,$$

with initial condition $u(x, y, 0) = u_0(x, y, 0)$, where $u$ denotes the populations density and $\mu$ is the Caputo fractional derivative operator.

### 2. Basic definitions of fractional calculus and Laplace Transform

In this section, we give some basic definitions and properties of fractional calculus and some properties of Laplace transform for fractional derivative which have been used in this paper:

**Definition 2.1** A real function $f(t), t > 0$ is said to be in the space $C_p, \mu \in R$ if there exists a real number $p > \mu$, such that $f(t) = t^\mu f_1(t)$ where $f_1(t) \in C(0, \infty)$ and it is said to be in the space $C_n$ if and only if $f^{(n)} \in C_\mu, n \in N$.

**Definition 2.2** The left sided Riemann-Liouville fractional integral operator of order $\mu \geq 0$, of a function $f \in C_\mu, \alpha \geq -1$ is defined as [28, 29]

$$I^\mu f(t) = \begin{cases} \frac{1}{\Gamma(\mu)} \int_0^t (t - \tau)^{\mu-1} f(\tau) d\tau, & \mu > 0, \ t > 0, \\ f(t), & \mu = 0 \end{cases},$$

(2.4)

where $\Gamma(\mu)$ is the well-known Gamma function.

**Definition 2.3** The left sided Caputo fractional derivative of $f, f \in C^n_\mu, m \in N \cup \{0\}$ is defined as

$$D^\mu f(t) = \frac{\partial^\mu f(t)}{\partial t^\mu} = \sum_{k=0}^{\mu-1} \frac{\partial^k f(t)}{\partial t^k}, \quad m - 1 < \mu < m, \ m \in N,$$

$$\mu = m,$$

(2.5)

(i) $I^\mu f(x, t) = \frac{1}{\Gamma(\mu)} \int_0^t \frac{f(x, t)}{(t-s)^{\mu}} ds, \quad \mu > 0, \ t > 0,$$

(ii) $D^\mu f(x, t) = \frac{\partial^\mu f(x, t)}{\partial t^\mu}, \quad m - 1 < \mu \leq m.$

**Definition 2.4** The Mittag-Leffler function $E_\alpha(z)$ with $\alpha > 0$ is defined by the following series representation, valid in the whole complex plane [31]:

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)},$$

(2.6)

**Definition 2.5** The Laplace transform of $f(t)$

$$F(s) = L[f(t)] = \int_0^\infty e^{-st} f(t) dt.$$  

(2.7)

**Definition 2.6** The Laplace transform of the Riemann-Liouville fractional integral [2] is defined as

$$L[I^\mu f(t)] = s^{-\mu} F(s).$$  

(2.8)

**Definition 2.7** The Laplace transform of the Caputo fractional derivative [2] is defined as

$$L[D^\mu f(t)] = s^\mu F(s) - \sum_{k=0}^{\mu-1} s^{\mu-1-k} f^{(k)}(0), \ n - 1 < \alpha \leq n.$$  

(2.9)

### 3. Homotopy Perturbation transform method

In order to elucidate the solution procedure of the homotopy perturbation transform method, we consider the following nonlinear fractional differential equation:

$$D^\mu u(x,t) + R(u(x,t)) + Nu(x,t) = g(x,t), \quad t > 0, \ x \in R, \ 0 < \alpha \leq 1, \ u(x,0) = h(x),$$

(3.10)

where $D^\mu = \frac{\partial^\mu}{\partial t^\mu}$ is the Caputo fractional derivative operator, $R$ is the linear operator, $N$ is the nonlinear operator and $g(x,t)$ is continuous function. Now, applying Laplace transform first on both sides of Eq. (3.10), we get

$$L[D^\mu u(x,t)] + L[R(u(x,t)) + Nu(x,t)] = L[g(x,t)],$$

(3.11)

Now, using the property of Laplace for fractional derivative as given by equation (2.9), we have
\( L[u(x,t)] = s^{-1}h(x) + s^{-\alpha}L[q(x,t)] - s^{-\alpha}L[Ru(x,t) + Nu(x,t)], \)

Operating the Inverse Laplace transform on both sides in Eq. (3.12), we obtain

\[ u(x,t) = G(x,t) - L^{-1}\left[s^{-\alpha}L[Ru(x,t) + Nu(x,t)]\right], \]

(3.13)

where \( G(x,t) \), represents the term arising from the source term and the prescribed initial conditions. Now, applying the homotopy perturbation technique, the solution can be expressed as a power series in \( p \) i.e.

\[ u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t), \]

(3.14)

where the homotopy parameter \( p \) is considered as a small parameter \( (p \in [0,1]) \). The nonlinear term can be decomposed as

\[ Nu(x,t) = \sum_{n=0}^{\infty} p^n H_n(u), \]

(3.15)

Substituting equation (3.14) and (3.15) in Eq. (3.13) and using HPM [11-17], we get

\[ \sum_{n=0}^{\infty} p^n u_n(x,t) = G(x,t) - \sum_{n=0}^{\infty} p^n H_n(u), \]

(3.16)

Now, equating the coefficient of like power of \( p \) on both sides, the following approximations are determined as follow

\[ p^0: u_0(x,t) = G(x,t), \]

\[ p^n: u_n(x,t) = \frac{1}{n!} \frac{\partial^n}{\partial p^n} \left[ \left( \sum_{n=0}^{\infty} p^n H_n(u) \right) \right], \quad n = 0, 1, 2, 3, \ldots \]

(3.17)

Proceeding in this same manner, the rest of the components \( u_n(x,t) \) can be completely obtained and the series solution is thus entirely found.

Finally, we approximate the solution \( u(x,t) \) by truncated series

\[ u(x,t) = \lim_{N \to \infty} \sum_{n=1}^{N} u_n(x,t), \]

(3.18)

The above series solutions generally converge very rapidly. A classical approach of convergence of this type of series is already presented by Abbaoui and Cherruault [32].

4. Numerical Examples

In this section, three examples of time-fractional biological equation are given to demonstrate the performance and efficiency of the HPM with coupling of Laplace transform. We have also study the solution profile of \( u(x,t) \) for different values of \( \alpha \).

Example 1: We consider the following time fractional biological population model

\[ \frac{\partial^\alpha u(x,y,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2} + h(x,y,t), \quad 0 < \alpha \leq 1, \]

(4.19)

subject to the initial condition \( u(x,y,0) = \sqrt{xy} \).

(4.20)

By applying the Laplace Transform and subject to initial condition (4.20) for equation (4.19), we have

\[ \frac{\partial^\alpha u(x,y,t)}{\partial t^\alpha} = \frac{1}{s^\alpha} L \left[ D_{xx} u^2 + hu \right], \]

(4.21)

where \( D_{xx} = \frac{\partial^2}{\partial x^2}, D_{yy} = \frac{\partial^2}{\partial y^2} \). and \( D_{xy} = \frac{\partial^2}{\partial x \partial y} \).

Taking the Inverse Laplace transform of (4.21), we have

\[ u(x,y,t) = \sqrt{xy} + \frac{1}{s^\alpha} L \left[ D_{xx} u^2 + hu \right]. \]

(4.22)

According to the homotopy perturbation method, we have constructed the homotopy

\[ \sum_{n=0}^{\infty} p^n u_n(x,y,t) = \sqrt{xy} + p L^{-1}\left[ \frac{1}{s^\alpha} L \left[ D_{xx} u^2 + hu \right] \right], \]

(4.23)

where \( H_n(u) \) are He’s polynomials that represent the nonlinear term.

Comparing the coefficient of like power of \( p \), we get

\[ p^0: u_0(x,y,t) = \sqrt{xy}, \]

(4.24)
\[ p_1 : u_1(x, y, t) = \sqrt{xy} \frac{h t^\alpha}{\Gamma(\alpha + 1)}, \]  
(4.25)

\[ p_2 : u_2(x, y, t) = \sqrt{xy} \frac{(h t^\alpha)^2}{\Gamma(2\alpha + 1)}, \]  
(4.26)

\[ p_3 : u_3(x, y, t) = \sqrt{xy} \frac{(h t^\alpha)^3}{\Gamma(3\alpha + 1)}, \]  
(4.27)

and so on.

Using the equations (4.24) – (4.27) in (4.23) we obtain
\[ u(x, y, t) = \sqrt{xy} \left( E_\alpha(h t^\alpha) \right), \]  
(4.29)

where \( E_\alpha(h t^\alpha) = \sum_{n=0}^{\infty} \frac{h^n t^{n\alpha}}{\Gamma(\alpha n + 1)} \).

As \( \alpha \to 1 \) then we have
\[ u(x, y, t) = \sqrt{xy} \ e^{ht}, \]  
(4.30)

which is the exact solution to the standard form.

The comparison between exact solution and approximate solution is shown through Fig.1 for the standard case i.e. \( \alpha = 1 \). Fig.2 show the solution profiles of \( u(x, t) \) for different values of \( \alpha \). It is shown through Fig.2 that as the value of \( \alpha \) decreases increase in \( u(x, t) \) is lesser.

**Example 2:** We have taken the time fractional biological population model in the following form
\[ \frac{\partial^\alpha u(x, y, t)}{\partial t^\alpha} = \frac{\partial^2 u(x, y, t)}{\partial x^2} + \frac{\partial^2 u(x, y, t)}{\partial y^2} + u(x, y, t), \quad 0 < \alpha \leq 1, \]  
(4.31)

the initial condition is \( u(x, y, 0) = \sqrt{\sin x \sinh y} \).

Using the Laplace Transform in (4.31) with initial condition (4.32), yields
\[ u(x, y, s) = \frac{1}{s} \sqrt{\sin x \sinh y} + \frac{1}{s^\alpha} \left[ \left( D_x + D_y \right) u^2 + u \right], \]  
(4.33)

where \( D_x = \frac{\partial}{\partial x}, \quad D_y = \frac{\partial}{\partial y}, \) and \( D_{xx} = \frac{\partial^2}{\partial x^2} \).

Further, using HPM and inverse Laplace transform in (4.33) we obtain
\[ \sum_{n=0}^{\infty} p^n u_n(x, y, t) = \sqrt{\sin x \sinh y} + p \mathcal{L}^\alpha \left[ \frac{1}{s^\alpha} \left( \sum_{n=0}^{\infty} p^n H_n(u) \right) \right], \]  
(4.34)

where \( H_n(u) \) are He’s polynomials.

Equating the like powers of \( p \), of equation (4.34), we have
\[ p_0 : u_0(x, y, t) = \sqrt{\sin x \sinh y}, \]  
(4.35)

\[ p_1 : u_1(x, y, t) = \sqrt{\sin x \sinh y} \frac{t^\alpha}{\Gamma(\alpha + 1)}, \]  
(4.36)

\[ p_2 : u_2(x, y, t) = \sqrt{\sin x \sinh y} \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)}, \]  
(4.37)

\[ p_3 : u_3(x, y, t) = \sqrt{\sin x \sinh y} \frac{t^{3\alpha}}{\Gamma(3\alpha + 1)}, \]  
(4.38)

and so on.

Using equations (4.35) – (4.38) in (4.34), yields
\[ u(x, y, t) = \sqrt{\sin x \sinh y} \sum_{n=0}^{\infty} \frac{t^{n\alpha}}{\Gamma(n\alpha + 1)}, \]  
(4.39)

Equation (4.39) converges to
\[ u(x, y, t) = \sqrt{\sin x \sinh y} \sum_{n=0}^{\infty} \frac{t^{n\alpha}}{\Gamma(n\alpha + 1)}, \]  
(4.40)

where \( \sum_{n=0}^{\infty} \frac{t^{n\alpha}}{\Gamma(n\alpha + 1)} \).

Our result is in complete agreement with [33].

As \( \alpha \to 1 \) then we have
\[ u(x, y, t) = \sqrt{\sin x \sinh y} \ e^t, \]  
which is the exact solution to the standard form.

The comparison between exact solution and approximate solution is shown through Fig.3 for the standard case i.e. \( \alpha = 1 \). Fig.4 show the solution profiles of \( u(x, t) \) for different values of \( \alpha \) which is completely reverse as in the case of the first example i.e. as the value of \( \alpha \) decreases increase in \( u(x, t) \) is more.

**Example 3:** Let us consider the following time fractional biological population model
\[
\frac{\partial^\alpha u(x,y,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2} - u(x,y,t) \left(1 + \frac{8}{9} w(x,y,t)\right), \quad 0 < \alpha \leq 1, 
\]
(4.41)

subject to the initial condition, \[u(x,y,0) = e^{x+y}.\]
(4.42)

By applying the Laplace Transform in equation (4.41) and using condition (4.42) we get
\[
u(x,y,s) = e^{x+y} + \frac{1}{s^{\alpha}} L \left[ \left(D_{xx} + D_{yy}\right) u^2 - u - \frac{8}{9} u^2 \right]
\]
(4.43)

where \[D_{xx} = \frac{\partial^2}{\partial x^2}, D_{yy} = \frac{\partial^2}{\partial y^2}.\]

Now, taking the Inverse Laplace Transform on equation (4.43) and applying HPM we have
\[
\sum_{n=0}^{\infty} p^n H_n(u) = e^{x+y} + \frac{1}{s^{\alpha}} L \left[ \sum_{n=0}^{\infty} p^n H_n(u) \right]
\]
(4.44)

where \[H_n(u)\] are He’s polynomials.

Now equating the terms with identical powers of \(p\) both sides we have
\[
p_0 : u_0(x,y,t) = e^{x+y}
\]
(4.45)
\[
p_1 : u_1(x,y,t) = -e^{x+y} \frac{t^\alpha}{\Gamma(\alpha + 1)}
\]
(4.46)
\[
p_2 : u_2(x,y,t) = e^{x+y} \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)}
\]
(4.47)
\[
p_3 : u_3(x,y,t) = -e^{x+y} \frac{t^{3\alpha}}{\Gamma(3\alpha + 1)}
\]
(4.48)
so on.

We obtained the solution in a series form as
\[
u(x,y,t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \left[ \frac{(-1)^n}{\Gamma(\alpha + 1)} + \frac{\Gamma(\alpha + 1)}{\Gamma(2\alpha + 1)} + \frac{\Gamma(\alpha + 1)}{\Gamma(3\alpha + 1)} + \cdots \right]
\]
(4.49)

It can be written in the closed form as
\[
u(x,y,t) = e^{x+y} E_{\alpha}(-t^\alpha).
\]
(4.50)

This is in complete agreement with [33].

If \(\alpha \rightarrow 1\) then
\[
u(x,y,t) = e^{\frac{1}{3}(x+y)} e^{-t},
\]
which is the exact solution for the standard form of the equation.

The comparison between exact solution and approximate solution is shown through Fig.5 for the standard case i.e. \(\alpha = 1\). It is shown through Fig.6 that for \(\alpha = 1\) the solution profiles of \(u(x,t)\) tend to zero as time increases. But if we decreases the value of \(\alpha\) then there is drastic change in the solution profile of \(u(x,t)\), i.e. it is not tending to zero but tending to infinity.

5. Conclusion

In this paper we have discussed the approximate solution of time fractional biological population model. We have employed a technique which is a combination of Laplace transform and homotopy perturbation method. Also we have taken three examples to discuss the biological model and its behavior for different values of \(\alpha\). The numerical procedure shows that method is simple and efficient even for solving fractional differential equations.

References


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Fig. 1 Plot of $u(x, y, t)$ for $t = 1, h = 0.2$ and $\alpha = 1$, Fig.1(a) Approximate Solution, 1(b) Exact Solution

Fig. 2 Plot of $u(x, y, t)$ versus $t$ for $x = 1, y = 1, h = 0.2$ and for different values of $\alpha$
Fig. 3 Plot of $u(x, y, t)$ for $t = 1$ and $\alpha = 1$. Fig. 3(a) Approximate Solution, Fig. 3(b) Exact Solution

Fig. 4 Plot of $u(x, y, t)$ versus $t$ for $x = 1, y = 1$ and for different values of $\alpha$
Fig. 5 Plot of $u(x, y, t)$ for $t = 1$ and $\alpha = 1$. Fig. 5(a) Approximate Solution, Fig. 5(b) Exact Solution.

Fig. 6 Plot of $u(x, y, t)$ versus $t$ for $x = 1, y = 1$ and for different values of $\alpha$. 