Number Of Level Crossings Of A Orthogonal Polynomial

DIPTY RANI DHAL, Dept. of Basic Science and humanities, ITER,BBSR, ODISHA,INDIA
DR.P.K.MISHRA, Dept. of Basic Science and humanities, CET,BPUT,BBSR, ODISHA,INDIA

Abstract
The aim of this paper is to estimate the number of real zeros of a Cauchy random polynomial under different conditions when the coefficients belong to the domain of attraction of Cauchy law. Let \( \sum_{r=0}^{n} G_r(X)W^r \) be a Cauchy random algebraic polynomial of degree \( n \), whose coefficients \( G_r(X)'s \) are identically distributed independent random variables belonging to the domain of attraction of Cauchy distribution. Then there exists a positive integer \( n_0 \) such that for \( n>n_0 \), the number of real roots of most of the equation \( \sum_{r=0}^{n} G_r(X)W^r = 0 \) is atmost \( \mu (\log n)^{\frac{1}{2}} \) except for a set of measure atmost \( n_0^{-1-s} \), where \( \mu \) and \( \mu' \) are positive constants and \( 0<s<1 \).


Keywords: Independent identically distributed random variables, random algebraic polynomial, random algebraic equation, real roots

Introduction
Let \( N_n(X) \) be the number of maxima & minima of the random algebraic polynomial \( \sum_{r=0}^{n} G_r(X)W^r \), where the coefficients \( y_r(u)'s \) are independent random variables. Samal [3] has considered the general case when \( G_r(X)'s \) are identically distributed normal variates with mean zero, variance one and third absolute moment finite and non-zero. Evans [1] has studied the strong version of the upper bound of \( N_n(X) \) (when the coefficients are normal variates). Samal and Mishra [4] have considered the upper bound \( N_n(X) \) (when the coefficients are identically distributed symmetric stable variables). But we give an estimate for an upper bound of \( N_n(X) \) when the coefficients are independent and identically distributed. Samal, Nayak and Pattnayak[2] consider the lower bounds of the number of real roots of a random algebraic polynomial and get a better result. In the following theorems we give an estimate for upper bound of the number of real zeros of a random polynomial, when the coefficients belong to the domain of attraction of Cauchy law both in strong and weak sense.

1.2. Theorem 1.
Let \( \sum_{r=0}^{n} G_r(X)W^r \) be a random algebraic polynomial of degree \( n \), whose coefficients \( G_r(X)'s \) are identically distributed independent random variables belonging to the domain of attraction of Cauchy distribution. Then there exists a positive integer \( n_0 \) such that for \( n>n_0 \), the number of real roots of most of the equation \( \sum_{r=0}^{n} G_r(X)W^r = 0 \) is atmost \( \mu (\log n)^{\frac{1}{2}} \) except for a set of measure atmost \( n_0^{-1-s} \), where \( \mu \) and \( \mu' \) are positive constants and \( 0<s<1 \).

1.3. Theorem 2.
Let \( \sum_{r=0}^{n} G_r(X)W^r \) be a random algebraic polynomial of degree \( n \), whose coefficients \( G_r(X)'s \) are identically distributed independent random variables belonging to the domain of attraction of Cauchy distribution. Then there exists a positive integer \( n_0 \) such that for \( n>n_0 \), the number of real roots of most of the equation \( \sum_{r=0}^{n} G_r(X)W^r = 0 \) is atmost \( \mu (\log n)^{\frac{1}{2}} \) except for a set of measure atmost \( n_0^{-1-s} \), where \( \mu \) and \( \mu' \) are positive constants and \( 0<s<1 \).

1.4. Corresponding to each root of \( f(x)=0 \) in \( (0,1) \). There exists a root of \( f(-x)=0 \) in \( (-1,0) \) and conversely. Again \( f(x)=0 \) has a root in \( (1,\infty) \) implies \( x^n f(G)=0 \).
has root in (0,1) where \( G = \frac{1}{x} \).

Sambadham [5] consider the number of real zeros of a random algebraic polynomial in the interval (0,1). The number of zeros in the interval \(( -\infty, \infty)\) and the measure of the exceptional set will be each four times of the corresponding estimates for the range (0,1). We choose a fixed number \( p \) greater than \( \frac{1}{\log 2} \) and let \( k = \lfloor p \log n \rfloor \), where \( \lfloor p \log n \rfloor \) denotes the greatest integer not exceeding \( p \log n \). We consider circles \( C_m, \ m = 1,2, \ldots, k \) with center 

\[
Z_0 = \sqrt{1+ \frac{2}{n}}.
\]

The characteristics function \( \Phi(t) \) of a distribution belonging to the domain of attraction of the Cauchy distribution is given by

\[
\Phi(t) = \exp \left( - C |t| h(t) \right)
\]

where \( h(t) \) is a slowly varying function as \( t \to \infty \)

or

\[
\Phi(t) = \exp \left( - C |t| H(t) \right)
\]

where \( H(t) \) is a slowly varying function as \( t \to 0 \).

1.5. We need the following lemmas for the proof of the theorem.

**Lemma 1.1**
If \( G(X) \) is identically distributed independent random variables belonging to the domain of attraction of the Cauchy distribution, then

\[
P \left\{ \left| G(X) \right| \leq \eta \right\} < \frac{2n}{\pi} \left( \frac{1}{1+1} \right)^{\frac{1}{1+2}} + \frac{\mu}{(n+1)^{s}} \left( \frac{1}{1-1} \right)^{\frac{1}{1+2}}
\]

**Proof:** This is the direct consequence of the paper Mishra, Nayak and Pattanayak [2] since the exponent \( \alpha \) of stable law takes value 1 in Cauchy law.

**Lemma 1.2**
If the random variable \( y(u) \) belong to the domain of attraction of Cauchy law, then for \( s > 0 \),

\[
P \left\{ \left| G(X) \right| \leq \eta \right\} \leq \frac{\mu}{\eta^{1-s}}
\]

1.6. **Proof of the theorem.**
Let us consider the circle \( C_m \) with \( Z_0 = x_m \), \( R = 2r_m \), \( r = \frac{r_m}{2n} \) and \( N_m(u) \) be the number of real zeros of

\[
\sum_{r=0}^{n} G_r(X) W^r.
\]

Then applying lemma 1.2 we have,

\[
N_m(X) < \frac{1}{\log 2} \log \left( \frac{z \max G(z)}{G(x_m)} \right)
\]

(1.2)

By using lemma 1.1 we have for \( s > 0 \),

\[
\begin{align*}
&\mathbb{P} \left\{ G_r(X) \geq (n+1)^s, 0 \leq r \leq n \right\} \\
&= \frac{\mu}{(n+1)^{2-s}} \text{ (for } s = 3s) \\
&\leq \frac{\mu}{(n+1)^{2-s}} \text{ (for } s = 3s)
\end{align*}
\]

Thus

\[
> 1 - \frac{\mu}{(n+1)^{2-s}}
\]

(1.4)

If \( G(z) \) is a regular function in a circle with center \( z_0 \) and of radius \( r' \), it follows from lemma 1.2 that outside a set a measure atmost

\[
\frac{\mu}{(n+1)^{2-s}}
\]

(1.5)

Now using lemma 1.1, for \( s > 0 \), we have

\[
\left| G_r(X) \right| \leq \frac{1}{n^{s}}
\]

(1.6)

except for a set of measure atmost
\[
\frac{2}{n^5} \left[ \frac{\Gamma\left(\frac{1}{\beta}\right)}{(\beta)^{C_{\beta}}} + \frac{\Gamma\left(\frac{1}{\nu}\right)}{\nu^{C_{\nu}}} \right] \tag{1.7}
\]

(putting \(\beta=1+s, \nu=1-s\))

Again it follows from above

\[
\frac{\mu}{n^s} \left\{ \sum_{r=0}^{n} x^{r_{\beta}} \left[ \sum_{r=0}^{n} x^{r_{\nu}} \right]^{-1/\beta} \right\} \tag{1.8}
\]

\[
\leq \frac{\mu}{n^s} \left\{ \left( \frac{1}{(n+1)^{1/\beta}} \right) + \frac{1}{(n+1)^{1/\nu}} \right\}
\]

\[
< \frac{\mu}{n^{s+1/\beta}} \tag{1.9}
\]

Therefore,

\[
|G(x_0)| \leq \frac{1}{n^5} \tag{1.10}
\]

except a set of measure atmost

\[
\frac{\mu}{n^{s+1/\beta}} \tag{1.11}
\]

Hence using \((1.4), (1.5), (1.9)\) we get from \((1.2)\) that outside a set of measure atmost

\[
\frac{\mu}{(n+1)^{2-s}} + \frac{\mu'}{n^{s+1/\beta}} \tag{1.12}
\]

The number of zeros of \(f(x)\) in \(C_m\) is atmost

\[
\log(\epsilon^2 (n+1)^{s}n^5) \leq \frac{\mu}{n^s} \leq \frac{\mu}{(n+1)^{2-s}} + \frac{\mu'}{n^{s+1/\beta}}
\]

except for a set of measure atmost

\[
\frac{\mu}{n^{s+1/\beta}} \tag{1.13}
\]

1.6. Let,

\[
L_i = \sum_{m=1}^{\log_2 \mu} \left( \sum_{r=0}^{n} \left( \frac{1 - 1}{2^n} \right)^r \right)^{-1/\beta} + \sum_{r=0}^{n} \left( \frac{1 - 1}{2^n} \right)^r \beta^{-1/\nu}
\]

w we have,

\[
= \sum_{m=1}^{\log_2 \mu} \left( \sum_{r=0}^{n} \left( \frac{1 - 1}{2^n} \right)^r \right)^{-1/\beta} + \sum_{r=0}^{n} \left( \frac{1 - 1}{2^n} \right)^r \beta^{-1/\nu}
\]

Again using the inequality \((1-x)^x < 1-nx\) when \(n < 0 \text{ or } n > 1\) for \(|x| < 1, r \neq 0\) we have

\[
\left( 1 - \frac{1}{2^m} \right)^\beta \geq \left( 1 - \frac{\beta}{2^m} \right)
\]

\[
\left( 1 - \frac{1}{2^m} \right)^\beta \geq \left( 1 - \frac{\beta}{2^m} \right) \tag{1.16}
\]

Therefore from \((1.15), (1.16)\) we have, in \(S_1\),

\[
\text{The number of zeros of } f(x) \text{ in } C_m \text{ is atmost }
\]

\[
\frac{\log(\epsilon^2 (n+1)^{s}n^5)}{\log 2} = \frac{2 + 4\log(n+1) + 5\log n\mu}{\log 2} = \mu \log n, \text{ for } m > 0.
\]

Again using \((1.4), (1.5), (1.11)\) we get from \((1.2)\) that outside a set of measure atmost
\[
\sum_{r=0}^{n} \left( 1 - \frac{1}{2^m} \right)^{\beta m} \geq 1 - \left( 1 - \frac{1}{n} \right)^{\beta n} \left( \frac{\beta}{2^m} \right)
\]

Now taking the 1st part of \( S_1 \),

\[
\log \frac{n}{\log n} \sum_{m=1}^{\log n} \left\{ \sum_{r=0}^{n} \left( 1 - \frac{1}{2^m} \right)^{\beta m} \right\}^{1/\beta} - 1
\]

< \sum_{m=1}^{\log n} \left\{ \frac{1 - \left( 1 - \frac{1}{n} \right)^n}{\beta / 2^m} \right\}^{1/\beta}

(Proceeeding similarly as 1st part of \( S_1 \))

So

\[
\frac{S_1}{n^2} < \frac{1}{n^3} (\mu_1 + \mu_2) = \frac{\mu n}{n^2} \quad (1.19)
\]

Now taking the 1st part of \( S_2 \)

\[
\sum_{\log n}^{\log n} \left\{ \sum_{r=0}^{n} \left( 1 - \frac{1}{2^m} \right)^{\beta m} \right\}^{1/\beta}
\]

< \sum_{\log n}^{\log n} \left\{ \left( n - \frac{1}{n} \right)^{\beta n} \right\}^{1/\beta}

\[
\leq \frac{\log n}{n^{1/\beta} e^{-1/D}} \quad \text{(where } D > 1)\]

< \frac{\mu}{\mu} \frac{\log n}{n^{1/\beta}} \quad (1.20)

Now taking the 2nd part of \( S_2 \)

So by the help of (1.19) and (1.22) we get from (1.13) that, the measure of exceptional set is atmost.

1.7. Now applying the procedure of Samal and Mishra [4] we consider the sequent \((0,1/2)\) we take a circle with center zero and radius \(1/2\).

The circle \(|z| \leq 1/2\) is interior to the circle \(|z| \leq 1\). Applying lemma 1.1 with \(z_0=0, r=1/2, R=1\), we have the number of zeros of \(f(x)\) in the circle \(C\) i.e.

\[
|z| \leq \frac{1}{2} \quad \text{does not exceed}
\]

\[
\frac{1}{\log 2} \log \left( \max_{|z| \leq 1} |f(z)| \right) \quad (1.24)
\]

we know from (4.3) that

\[
P \{ y, (u) \leq (n+1)^3, 0 \leq r \leq n \} > 1 - \frac{\mu}{(n+1)^{2-s}}
\]

So outside a set of measure atmost

\[
\frac{\mu}{(n+1)^{2-s}} \quad (1.25)
\]

We have,

\[
\max_{|z| \leq 1} |f(z)| \leq \max_{|z| \leq 1} \sum_{r=0}^{n} |y, (u)||z|^r
\]

\[
\leq \sum_{r=0}^{n} (n+1)^3 1^r
\]

\[
\leq \mu(n+1)^4 \quad (1.26)
\]

By using lemma 1.1 we obtain that

\[
P \left\{ |f(o)| \leq \frac{1}{n^5} \right\} < \frac{\mu \log n}{n^{2-s}} \quad (1.27)
\]

[Since \(2-s<5\), for \(s>0\) and \(\log n<n/4 \leq \log n(n)^{2-s}\)]

So using (1.25), (1.26), (1.27) we get from (1.24) that the number of zeros inside the circle \(C\) does not exceed
\[
\frac{\log \mu n^9}{n^2} < \mu \log n < \mu (\log n)^2\quad (1.28).
\]

Outside a set of measure atmost
\[
\frac{\mu}{(n+1)^2} + \frac{\log n}{n^{2-\varepsilon}} < \frac{\mu \log n}{n^{2-\varepsilon}}\quad (1.29)
\]

1.8 In case \(y_0(u)\) and hence \(f(o)\) is zero with positive probability, we take a circle with center \(0 < \sigma < \frac{1}{2}\) and radius \(\frac{1}{2}\). Thus the circle \(|z - \sigma| \leq (1 - \sigma)\) is interior to the circle \(|z| \leq (1)\).

Now considering the procedure of Samal and Mishra [4]

\[
\sigma = 1, r = \frac{1}{2}, R = (1-0),
\]

and applying lemma 1.1 with \(n=2, n=3\), the number of zeros of \(f(x)\) in the circle \(C''\), i.e. \(|z - \sigma| \leq (1/2)\) does not exceed

\[
\log \max_{|z| \leq 1} \left| \frac{f'(z)}{f'(0)} \right| \quad (1.30)
\]

We know from (1.25) and (1.26) that outside a set of measure atmost

\[
\frac{\mu}{(n+1)^2} < \frac{\mu}{n^{2-\varepsilon}}\quad (1.31)
\]

\[
\max_{|z| \leq 1} |f(z)| \leq 7(n+1)^4\quad (1.32)
\]

By using lemma 1.2, we have

Therefore

\[
|f'(\sigma)| \geq \frac{1}{\mu^{2-\varepsilon}},
\]

except for a set of measure atmost

\[
\frac{\mu \log n}{n^{2-\varepsilon}}\quad (1.34)
\]

So by using (1.31), (1.32), (1.34) we obtain from (1.30) that the number of zeros inside the circle \(C''\) does not exceed

\[
\frac{\log \mu n^9}{n^2} < \mu \log n < \mu (\log n)^2\quad (1.35)
\]

Outside a set of measure atmost

\[
\frac{\mu}{(n+1)^2} + \frac{\mu \log n}{n^{2-\varepsilon}} < \frac{\mu \log n}{n^{2-\varepsilon}}\quad (1.36)
\]

1.9 Now let us consider the number of real zeros \(N_n\) in the whole interval \((0,1)\). From the sections (1.5), (1.7), (1.8) that \(N_n < (\log n)^2\).

Outside a set of measure atmost

\[
\sum_{n=0}^{\infty} \frac{\mu' \log n}{n^{2-3s}} < \frac{\mu'}{n^{2-3s}}\quad (1.37)
\]

(since for large \(n\), \(\log n < n^2s\))

\[
\mu' < \frac{1-s}{n\log 2}, \quad \text{where} \quad 1 > s > 0.
\]

1.10 Proof of the theorem 1.2.

We cover the closed segment \([1/2, 1]\) by the circles mentioned in the section (1.3), we have already established in (1.3) that

\[
P \left( \frac{y_0(u)}{(n+1)^3} : 0 \leq r \leq n \right) > 1 - \frac{\mu}{(n+1)^{2s}}\quad (1.37)
\]

Therefore except a set of measure atmost

\[
\frac{\mu}{(n+1)^{2s}}\quad (1.38)
\]

We have

\[
\max_{|z| \leq 1} |f(z)| \leq \sigma^2 (n+1)^4\quad (1.39)
\]

By using lemma (1.1), we get that

\[
\beta = 1 + s, \nu = 1 - s
\]
\[ \left\{ \left| y_r(u) \right| \geq \frac{1}{n} \right\} \]

or,

except for a set of measure atmost

(1.41)

So

\[ \frac{1}{n} \mu \]

except for a set of measure atmost

for \( m=1,2,3,\ldots,k, p \log n \).

Similarly

[Since \( \frac{1}{\beta} < \frac{1}{\nu} \)]

or

\[ \left| f(x_m) \right| \geq \frac{1}{n} \]

except for a set of measure atmost

\[ \frac{\mu}{n^{1+1/\beta}} \]  

(1.45)

By using (1.38), (1.39), (1.43) we get from (1.2) that outside a set of measure atmost

The number of zeros of \( f(x) \) in \( C_m \) is atmost

\[ \frac{\log \left( e^2(n + 1)^3 n \right)}{\log 2} < \mu \log n, \text{for } n > 0 \]

Using (1.38), (1.39), (1.44) we get from (1.2) that outside a set of measure atmost

\[ \left\{ \frac{\mu}{(n + 1)^{2-s}} + \frac{\mu}{n^{1+1/\beta}} \right\} \]

The number of zeros of \( f(x) \) in \( C_0 \) is atmost \( \mu \log n \), for \( m=0 \). Considering all the circles \( C_0, C_1, C_2,\ldots,C_k, C_p \) log n the total number of zeros inside all the circles is atmost.

\[ \mu (k + 2) \log n < (\log n)^2 \]  

(1.46)

except for a set of measure atmost

\[ \frac{\mu}{n^{1-s}} \]

(1.47)

1.11. Consider the sum

\[ \frac{1}{n} (S_1 + S_2) \]

Following the procedure adopted in section 1.6, we get

Therefore from (1.47) we obtain that the measure of the exceptional set is atmost (Since \( \log n < n^s \) for large \( n, 0<s<1 \))

1.12

It has been calculated in theorem 1.1, section 1.1.7 and 1.1.8 that the number of zeros in the segment \( (0,1/2) \) does not exceed \( \mu (\log n)^2 \). Outside a set of measure atmost

\[ \frac{\mu}{n^{1-s}} \]

when is less than \( \frac{\mu}{n^{1-s}} \).

Therefore considering the number of real zeros in the whole interval \( (0,1) \)

\[ N_n < \mu (\log n)^2. \]

Outside a set of measure atmost where \( s>0 \).

References


