Expected Number of Level Crossings of a Class of Algebraic Polynomial

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Abstract
This paper provides the expected number of $K$ level crossings of a class of random algebraic polynomial of the form

$$f(t) = \sum_{k=1}^{n} \tau_{k} t^{k}$$

Let $f(t) = \sum_{k=1}^{n} \tau_{k} t^{k}$ be a random algebraic polynomial. Then the average number of level crossings of the polynomial $f(t)=K$ is asymptotic to

$$\frac{1}{\pi} \log \left( \frac{n}{K} \right) \text{ in } (-1,0) \text{ and } (0,1);$$

(i).

$$\frac{1}{2\pi} \log n \text{ in } (-\infty,1) \text{ and } (1,\infty).$$

(ii).

Hence $K$ is any constant such that $K/n$ tends to zero as $n \to \infty$.


Keywords: Independent identically distributed random variables, random algebraic polynomial, random algebraic equation, real roots.

Theorem 1: Let $\{a_{k}\}$ be a sequence of martingale differences with $E(a_{k})=0$,

$$E(T_{i}^{2}) \sim \Delta^{2}; \quad \left(\Delta^{2} > 0, n \to \infty\right)$$

$$E(a_{k}^{2}) \leq M, \quad \text{ (for some } \delta > 0 \text{ and } M < \infty)$$

$$\tau_{k} = a_{(k-1)+1} + a_{(k-1)+2} + \cdots + a_{(k),}$$

$$T_{i} = n^{-1/2} \tau_{i}, \quad (k=1,2,\ldots)$$

and

$$E\left[ \sum_{i=1}^{n} |E(a_{k} \downarrow P_{k}) - E(a_{k}^{2})| \right] \leq B(n) \downarrow 0.$$}

Here $P_{k}$ are the past events generated by random variables $\{a_{k,1}, a_{k,2},\ldots\}$.

$$\frac{1}{\pi} \log n \text{ in } (-\infty,1) \text{ and } (1,\infty).$$

(ii).

Hence $K$ is any constant such that $K/n$ tends to zero as $n \to \infty$.

Introduction

Kac [36], [37] some years back studied the number of real zeros of polynomial of the form

$$f(t) = \sum_{k=1}^{n} a_{k} t^{k} = 0$$

$$f(t) = a_{k} t^{k} = 0$$

where $(a_{0}, a_{1}, \ldots, a_{n})$ is a sequence of independent normally distributed random variables with mathematical exception zero and variance unity. He found that the average number of real zeros of a polynomial of this form in the interval $(-\infty, \infty)$ is asymptotic to $2/\pi \log n$.

From the work of Ibragimov and Masolova [31] we find that if the coefficients $a_{i}$'s are independent distributed random variables belonging to the domain of attraction of normal law and have zero mean then the same asymptotic relation holds good.

Farahmand [20] considered a polynomial of the form $f(t)=K$ and found that as long as $K^{2}/n$ tends to zero for large $n$, the expectation of number of real zeros of the
above curve is asymptotic to
\[ \frac{1}{\pi} \log \left( \frac{n}{K^2} \right) \]
in the intervals \((-1,1)\) and \((1,\infty)\) turn out to be
\[ \frac{1}{2\pi} \log n \].

Sambadham [58] estimated the average number of real zeros of a random polynomial with hyperbolic elements;
\[ \sum_{k=1}^{n} \tau_{kn} \cosh kn, \]
i.e.
where \((a_k)\) is a sequence of martingale differences
\[ E(a_k)=0, \quad (1.1) \]
\[ E(t^2k)\sim A^2, \]
\[ \|E\|^{1/2} \leq M \quad (\text{for some } \delta < 0 \text{ and } M < \infty). \]

Where
\[ \tau_{kn}=a_{k-1}+a_{k-2}+\ldots+a_k, \]
\[ T_k=n^{-1/2} \tau_{kn} \quad (k=1,2,\ldots,n) \]
and
\[ \lim_{n \to \infty} \left| \sum_{k=1}^{n} E(a_k/P_k) - E(a_k^2) \right| \leq B(n) \downarrow 0. \quad (1.2) \]

Here \(P_k\) is the past event generated by the set of random variables \(\{a_{k-1}, a_{k-2},\ldots\}\).

Here we consider an algebraic polynomial of the form
\[ f(t) = \sum \tau_{kn} t^k = K, \quad (1.3) \]

Where the sequence of random variables \((a_k)\) satisfies the conditions \((1.1)\) and \((1.2)\) and \(K\) is any constant such that \(K/n \to 0\) as \(n \to \infty\). We represent by
\[ EN_s(\alpha, \beta) \]
the expected number of real zeros of equation \((1.3)\) in the interval \((\alpha, \beta)\).

Proof Of The Theorem
In 1.2 we have derived the extended Kac-rice formula to be used in the proof of the theorem.

In 1.3, 1.4, 1.5 and 1.6 respectively we deal with the intervals \((0,1)\), \((1,\infty)\), \((-1,0)\) and \((-\infty,1)\) to find the asymptotic estimates of average number of roots.

1.2. Formula for \( EN_s(\alpha, \beta) \)

Let \((a_k)_{k\in N}\) be a sequence of martingale differences (md). If in this case correlations exists, then the random variables are uncorrelated. This shows that each \(\tau_{kn}\) satisfies the condition \((1.1)\) and \((1.2)\), from Serfling [72] we find that \(\tau_{kn}\) is asymptotically normal with mean zero and variance \(n\) if \(E(a_k^2)=1\). Hence \(f(t)\) is asymptotically normal. For our equation \(f(t)-K=0\), it follows from Crammer and Leadbetter [7, p-285] that
\[ EN_s(\alpha, \beta) = \frac{1}{\sqrt{2\pi}} \left| \left| 1-\lambda^2 \right| \right| \Phi \left( \frac{m}{\sqrt{X}} \right) \left[ 2\Phi(\eta)-\eta(2\Psi(\eta)-1) \right] \cdot dt, \]
Where
\[ X = \text{var} \{f(t)-K\}, \]
\[ Z = \text{var} \{f'(t)\}, \]
\[ \lambda = (XZ)^{-1/2} \text{cov} \{f(t)-K, f'(t)\}, \]
\[ m = E\{f(t)-K\}, \]
\[ \eta = Z^{-1/2} (1-\lambda^2)^{-1/2} \left\{ \gamma - \lambda m(Z/X)^{1/2} \right\}, \]
\[ \gamma = E\{f'(t)\}. \]

\(\Phi\) and \(\Psi\) are standard density function and distribution functions respectively.

From these values we get
\[ m = -K, \]
\[ \gamma = 0, \]
\[ X = X(t) = n \sum_{k=0}^{n} t^{2k}, \]
\[ Z = Z(t) = n \sum_{k=1}^{n} k^{2} t^{2k-2}, \]
\[ Y = Y(t) = n \sum_{k=1}^{n} K t^{2k-1}, \]
\[ \lambda = -Y / \sqrt{XZ}, \]
\[ \eta = -\frac{YK}{\sqrt{XZ}}. \]

Hence from (1.4),
\[ \Psi(t) = \frac{1}{2} + \pi^{-1/2} \text{erf} \left( \frac{t}{\sqrt{2}} \right), \]

We have the extended Kac-Rice formula
\[
E_{X} \left( \alpha, \beta \right) = \left[ \frac{\Delta^{1/2}}{\pi X} \exp \left( \frac{-ZK^{2}}{2\Delta} \right) \right] + \left[ \frac{K^{2}}{\pi X^{1/2}} \exp \left( \frac{K^{2}}{2X} \right) \text{erf} \left( \frac{K}{\sqrt{2X}} \right) \right] dt
\]
\[ = \int_{a}^{b} I(t) dt = \int_{a}^{b} I_{1}(t) dt + \int_{a}^{b} I_{2}(t) dt, \quad (1.5) \]

Where
\[ I_{1}(t) = \frac{\Delta^{1/2}}{\pi X} \exp \left( \frac{-ZK^{2}}{2\Delta} \right) \]
\[ I_{2}(t) = \left( \frac{|K|Y\sqrt{2}}{\pi X^{1/2}} \right) \exp \left( \frac{K^{2}}{2X} \right) \text{erf} \left( \frac{|K|Y}{\sqrt{2X}} \right). \]

**Excepted number of roots in (0,1)**

For \( t \in (0,1) \) we have the following estimates

\[ X = \frac{n^{1/2} t}{1-t^{2}}, \quad (1.6) \]
\[ Y = n \left[ (1-t^{2})^{2} - n^{2} (1-t^{2}) \right] \quad (1.7) \]
and
\[ Z = n \left[ (1-t^{2})^{2} - 2n^{2} (1-t^{2}) \right]. \quad (1.8) \]

From (1.6), (1.7) and (1.8), we have
\[ \Delta = \Delta(t) = \frac{n^{3} \left[ (1-t^{2})^{2} - n^{2} (1-t^{2}) \right]}{(1-t^{2})^{4}} \quad (1.9) \]

Hence
\[ Z = \frac{(1-t^{2})^{2} - 2n^{2} (1-t^{2}) + (1+t^{2}) \left[ (1-t^{2})^{2} \right]}{n \left[ (1-t^{2})^{2} - n^{2} (1-t^{2}) \right]} \]
\[ \Delta \quad (1.10) \]

For large \( n \), we have
\[ t^{2n-2} \left( 1-t^{2} \right) ^{2} \leq \frac{4}{n^{2} e^{2}} \]

Hence from (1.10)
\[ \frac{Z}{\Delta} \quad (1.11) \]

By [36]
\[ \frac{\Delta^{1/2}}{\pi X} = \frac{\left[ (1-t^{2})^{2} \right]^{2}}{1-t^{2}} \quad (1.12) \]

Also for all \( 0 \leq t \leq 1 \), from (1.12), we have
\[ \frac{\Delta^{1/2}}{X} < (2\pi-1)^{1/2} (1-t)^{1/2} \] and
\[ \frac{\Delta^{1/2}}{X} < (1-t)^{-3}. \quad (1.13) \]

Thus
\[ \int_{0}^{1} \frac{\Delta^{1/2}}{X} \exp \left( \frac{-KZ}{2\Delta} \right) dt \]
\[
Y \leq nt(1-t^{2n})(1-t^2)^{-2}
\]
so that
\[
\frac{Y}{X^{3/2}} \leq n^{-1/2} t(1-t^{2n})^{-1/2} (1-t^2)^{-1/2}
\]
\[
\leq n^{-1/2} t(1-e^{2})^{-1/2} (1-t^2)^{-1/2}
\]
For \(1-1/n \leq t \leq 1\), we have
\[
Y \leq \frac{n^2}{t} \sum_{k=0}^{n} t^{2k}.
\]
Hence in this range of \(t\), for sufficiently large \(n\)
\[
\frac{Y}{X^{3/2}} \leq \frac{n^{1/2}}{t} (1-t)^{-1/2} (1-t^{2n})^{-1/2}
\]
\[
\leq \frac{n^{1/2}}{t} \| (1-1/n)^{1/2} \| (1-1/n)^{2k} \| ^{1/2}
\]
\[
\leq 2 \| (1-e^{-2})^{1/2} \| . 
\]
(1.16)
Also
\[
\int_{0}^{1} I_{t}(t) dt = \int_{0}^{1/4} I_{t}(t) dt + \int_{1/4}^{1} I_{t}(t) dt.
\]
Since \(\text{erf} (t) < 1\)
\[
\int_{0}^{1/4} I_{t}(t) dt \leq K \sqrt{2 \pi} (1-e^{-2})^{1/2} n^{1/2} \int_{0}^{1/4} (1-t^{1/2})^{1/2} \exp \left[ \frac{K^2 (1-t^4)}{n (1-t^{2n})} \right] dt
\]
\[
\leq K \sqrt{2 \pi} (1-e^{-2})^{1/2} n^{1/2} \int_{0}^{1/4} (1-t^{1/2})^{1/2} \exp \left[ \frac{K^2 (1-t^2)}{n (1-t^{2n})} \right] dt
\]
\[
\leq 2 \sqrt{2} \| (1-e^{-2})^{1/2} \| .
\]
(1.17)
\[
\text{after substituting} \quad \frac{K^2}{n} \quad \text{by} \quad u' \]
Since
From (1.14), (1.15), (1.17) and (1.18) it is observed that
\[ EN_n(0,1) < \frac{\pi^{-1}}{\log(n/k)} + 0(1). \]  
(1.19)

Now we proceed to obtain a lower bound for \( EN_n(0,1) \). For \( 0 \leq t \leq 1 - 1/n \), from (1.10), we have
\[
\frac{1}{\Delta} \left[ 1 - h_n(t) \right] \leq \frac{2(1-t^2)}{n^2(1-t^2)^2 - 4t^2} < C_1 \frac{(1-t^2)}{n},
\]
(1.20)

Where \( C_1 \) is a constant.

From (1.12), we have
\[
{\Delta}^{1/2} = \frac{\left(1 - h_n(t)^2\right)^{1/2}}{1 - t^2},
\]
\[
h_n(t) = \frac{nt^{n-1} - (1-t^2)}{1-t^{2n}}.
\]

For \( 0 \leq t \leq 1 - 1/n \),
\[
h_n(t) = nt^{n-1} \leq n(1 - n^{-1})^{n-1}
\]
and hence \( h_n(t) \) can be made smaller than \( e^{1/2} \) for sufficiently large \( n \) and \( e > 0 \).

Hence, for large \( n \), we have
\[
\Delta^{1/2} > \frac{(1-e)^{1/2}}{1-t^2} > \frac{(1-e)^{1/2}}{2(1-t)}.
\]
Taking \( p = C_1 K^2 \) and \( x = (1-t) \), we get
\[
EN_n(0,1) \geq \int_0^1 \frac{{\Delta}^{1/2}}{X} \exp\left(-\frac{K^2Z}{2\Delta}\right) dt
\]
\[
\geq (2\pi)^{1/2} \int_{1/2}^1 x^{-1} \exp(-px) dx
\]
\[
= (2\pi)^{1/2} \log n - (2\pi)^{1/2} \int_0^{\log(p)/p} \frac{(1-e^{-t})}{x} dt
\]
\[
+ (2\pi)^{1/2} \int_0^{\log(p)/p} \frac{(1-e^{-t})}{x} dt
\]
(1.21)

Since \( (p/n) \to 0 \)
\[
\int_0^{\log(p)/p} \frac{(1-e^{-t})}{x} dt = p/n + O\left(\frac{p^2}{n^2}\right)
\]
(1.22)

and
\[
\int_0^{\log(p)/p} \frac{(1-e^{-t})}{x} dt + \frac{(1-e^{-t})}{x} dt < \log(p) + 1.
\]
(1.23)

From (1.21), (1.22) and (1.23), we see that
\[
EN_n(0,1) \geq (\pi^{-1}) \log \left(\frac{n}{K}\right) - C_2,
\]
(1.24)

(for large \( n \))

Where \( C_2 \) is a constant.

From (1.19) and (1.24), we obtained
\[
EN_n(0,1) \sim (\pi^{-1}) \log \left(\frac{n}{K}\right)
\]
(1.25)

**Expected number of real zero in \((1,\infty)\)**

We first find an upper estimate of \( EN_n(1,\infty) \).

For \( 1<t<\infty \), we get
\[ Y < \frac{n^2}{t} \sum_{i=1}^{\infty} t^{2i} = \frac{n^2 (t^{2n} - 1)}{t(t^2 - 1)}. \]

Hence
\[ Y < \frac{n^2}{X^{3/2}} \left\{ \frac{n^{3/2}(t^{2n} - 1)(t^2 - 1)^{3/2}}{n^{3/2}(t^{2} - 1)(t^{2n} - 1)^{3/2}} \right\} \]
\[ = \frac{n^{1/2}(t^2 - 1)^{1/2}}{(t^{2n} - 1)^{1/2}}. \]

Putting \( t = 1/x \), we get
\[ Y < \frac{n^{1/2}X^2(1-x^2)^{1/2}}{(1-x^{2n})^{1/2}} \]

Also
\[ \leq \frac{K}{\sqrt{2\pi}} \int_{X^{3/2}}^{\infty} \exp \left\{ -\frac{Y}{2X} \right\} \text{erf} \left\{ \frac{\sqrt{2\lambda\Delta}}{X} \right\} dx \]
\[ \leq \frac{K}{\sqrt{2\pi}} \int_{X^{3/2}}^{\infty} \frac{Y(x)}{X^{3/2}} \left\{ \frac{n^{1/2}x^3}{n} \right\} dx \]
\[ + \frac{K}{\sqrt{2\pi}} \int_{X^{3/2}}^{\infty} \exp \left\{ -\frac{\sqrt{n}}{X(n-2)} \right\} \left\{ n \left( \frac{n-1}{n} \right) \right\}^{1/2}. \]

From (1.9) and (1.6), we obtain
\[ \Delta = n^2 x^{-(4n-8)} \left\{ 1 - h(x) \right\} (x^{2n-1})(x^2 - 1)^{-4}, \]
(1.29)

Where \( h(x) \) is defined previously and

\[ X = nx^{-2(\Delta_n)} (x^{2n-1})(x^2 - 1)^{-1}. \]
(1.30)

Hence from (1.13), (1.29) and (1.30)

\[ \int_{1}^{\infty} \frac{\Delta^{1/2}}{X} \exp \left\{ -\frac{YK^2}{(2\Delta)} \right\} dt \]
(1.26)
\[ < \int_{1}^{\infty} \frac{\Delta^{1/2}}{X} dt \]
(1.27)
\[ = \int_{0}^{1} \frac{\Delta^{1/2}}{X} x^{-3} \]
\[ = \int_{0}^{1} \frac{\Delta^{1/2}x^{-3} \exp \left\{ -\frac{Y}{x} \right\} dx + \int_{1}^{\infty} \frac{\Delta^{1/2}}{X} \exp \left\{ -\frac{Y}{x} \right\} dx \]
\[ < \int_{0}^{1} \left( 1 - x^3 \right) dx + \int_{1}^{\infty} \left( (2n-1) \right)^{1/2} (1-x)^{-1/2} \]
\[ < \frac{1}{2} \log n + o(1). \]
(1.31)

Hence

\[ EN_n(1, \infty) < \frac{1}{2\pi} \log n + C_3 \]
(1.32)

Where \( C_3 \) is a constant.

Now we obtain a lower estimate of \( EN_n(1, \infty) \)

From the estimates of \( z \) and \( \Delta \), from (1.8) and (1.9) respectively, in the range of
\[ 0 \leq x \leq 1 - \frac{1}{n} \]
\[ Z = \frac{x^{2n-4}(1-x^2)(1+x^2)(1+x^{n}) + n^2(1-x^2)^2 - 2n(1-x^2)}{n^2(1-x^2)} \]
\[ \frac{\Delta}{\Delta} \frac{n^2(1-x^2)(1+x^2)(1+x^{n}) + n^2(1-x^2)^2 - 2n(1-x^2)}{n^2(1-x^2)} \]
\[ (as \ h(1/1) = 1) \]
\[ < \frac{x^{2n-4}(1-x^2)(1+x^2)(1+x^{n}) + n^2(1-x^2)^2 - 2n(1-x^2)}{n^2(1-x^2)(1+x^2)(1+x^{n}) + n^2(1-x^2)^2 - 2n(1-x^2)} \]
\[ < \frac{x^{2n-4}(1-x^2)(1+x^2)(1+x^{n}) + n^2(1-x^2)^2 - 2n(1-x^2)}{n^2(1-x^2)^2 - 2n(1-x^2)} \]
\[ < \frac{x^{2n-4}(1-x^2)(1+x^2)(1+x^{n}) + n^2(1-x^2)^2 - 2n(1-x^2)}{n^2(1-x^2)^2 - 2n(1-x^2) + 1/n} \]
\[ < \frac{x^{2n-4}(1-x^2)(1+x^2)(1+x^{n}) + n^2(1-x^2)^2 - 2n(1-x^2)}{n^2(1-x^2)^2 - 2n(1-x^2) + 1/n} \]
\[ < \frac{x^{2n-4}(1-x^2)(1+x^2)(1+x^{n}) + n^2(1-x^2)^2 - 2n(1-x^2)}{n^2(1-x^2)^2 - 2n(1-x^2) + 1/n} \]
< 5nx^{2n-4}(1-x^2)^3, \quad (1.33)

For sufficiently large n.

Also for 0 \leq x \leq 1 - 1/n

x^{n-4}(1-x^2)^2 \leq \frac{9}{n^2e^2}.

Let s'' = \frac{9K^2}{ne^2}

Using the result of Kac [36], we have

$$\int_{X}^{\Delta^{1/2}} \left\{ \frac{ZK^2}{2X} \right\} dx$$

$$\geq \int_{0}^{1-1/n} (1-x)^{-1} \exp\left[-S''e^2n^2x^{2n-4}(1-x^2)^{1/18}\right] dx$$

$$\geq \int_{0}^{1-1/n} (1-x)^{-1} \exp\left[-S''x^n(1-x^2)/2\right] dx$$

$$\geq \frac{1}{2} \int_{0}^{1-1/n} (1-x)^{-1} \exp\left[-S''x^n(1-x)\right] dx. \quad (1.34)$$

For 0 \leq x \leq 1 - 1/n

\exp\left\{ -S'' x^n(1-x) \right\}

= 1 - S'' x^n(1-x) + O(S''^2/e^2n^2).

Also in this range

\max \left\{ x^n(1-x) \right\} < \frac{1}{en}.

Hence from (7.34), we obtain

\begin{align*}
\int_{X}^{\Delta^{1/2}} \exp\left\{ -\frac{ZK^2}{2\Delta} \right\} dx \\
\geq \frac{1}{2} \int_{0}^{1-1/n} (1-x)^{-1} dx - \frac{1}{2} \int_{0}^{1-1/n} S'' x^n dx \\
+ \frac{1}{2} \int_{0}^{1-1/n} O(S''/n^2) dx
\end{align*}

= \frac{1}{2} \log n + O(S''/n)

= \frac{1}{2} \log n + O(K^2/n^2)

Hence

$$\mathcal{E}(1,\infty) \geq \frac{1}{2\pi} \log n + O(K^2/n^2) \quad (1.35)$$

From (7.31) and (7.35), we have

$$\mathcal{E}(1,\infty) \sim \frac{1}{2\pi} \log n. \quad (1.36)$$

1.5. Expected number of real zeros in (-1,0)

Let y = t

Then for 0 < y < 1 - 1/n

$$\frac{K^2}{X(t)} = \frac{K^2(1-y^2)}{1-y^n} > K^2(1-y^2),$$

$$Y(t) = -ny^{2n+1} + y^{2n+1} + ny^{2n-1} - y$$

and

Now

$$\int_{\{X(t)^{1/2}\}}^{0} \exp\left\{ -\frac{K^2}{2X(t)} \right\} dt$$

$$= \int_{\{X(t)^{1/2}\}}^{0} \left( -ny^{2n+1} + y^{2n+1} \right) \exp\left\{ -\frac{K^2}{2X(y)} \right\} dy$$

$$\leq \int_{\{X(t)^{1/2}\}}^{0} \left( -ny^{2n+1} + y^{2n+1} - y \right) \exp\left\{ -\frac{K^2}{2X(y)} \right\} dy$$

Then using the procedure described in (1.17) and (1.18), we have

$$\int_{\{X(t)^{1/2}\}}^{0} \exp\left\{ -\frac{K^2}{2X(t)} \right\} dt$$

$$\leq 2\sqrt{\pi} \left( 1-\epsilon^{-2} \right)^{-1/2} + \frac{2\sqrt{\pi}}{\pi} \left| K \right| n^{-1/2} \left( 1-\epsilon^{-2} \right)^{-1/2} \exp\left\{ -\frac{K^2}{2n^2} \right\}. \quad (1.37)$$
Procedure similar to that shown in (1.14) shows that
\[
0 \leq \int_{-1/n}^{1/n} I_n(t) dt = \int_{0}^{1/n} I_n(y) dy \leq \log \left( \frac{n}{k} \right) + O(1).
\]
Also from (1.15), we have
\[
\int_{0}^{1/n} \frac{1}{t} \exp \left\{ - \frac{Z(t)K^2}{2\Delta(t)} \right\} dt \leq \sqrt{2} \left( 2 - \frac{1}{n} \right)^{1/2}.
\]
Hence from (1.37), (1.38) and (1.39), we have
\[
\text{EN}_n(-1,0) \leq \frac{1}{\pi} \log \left( \frac{n}{k} \right) + O(1).
\]
From the previous discussions, we have also
\[
\text{EN}_n(-1,0) > \int_{-1/n}^{1/n} I_n(t) dt = \int_{0}^{1/n} I_n(y) dy > \pi^{-1} \log \left( \frac{n}{k} \right) + O(1)
\]
(1.40)
1.6. Expected number of zeros in (-∞,1)

In order to evaluate \(\text{EN}_n(-\infty,-1)\), we let \(y=-1/t\).

Then
\[
\int_{-\infty}^{-1/n} I_n(t) dt = \int_{0}^{1/n} I_n(y) dy = \sqrt{\pi \Delta(-1/y)} \exp \left\{ \frac{Z(-1/y)K^2}{2\Delta(-1/y)} \right\} \frac{dy}{y^{3/2}}.
\]
Now
\[
\Delta(-1/y) = \frac{1}{y^{4n-4}(1-y^2)} \left\{ \frac{(1-y^2)^2}{(1-y^2)^2} - n^2 y^{2n-2} \right\}
\]
(1.38)
\[
\frac{ny^{n-1}}{1-y^2} \to 0 ; \text{ as } n \to \infty,
\]
(1.41)
But
\[
\Delta(t) = \Delta(-1/y) > \frac{(1-y^2)^2}{y^{4n-4}(1-y^2)} (1-\varepsilon)
\]
(1.42)
Since
\[
\frac{\sqrt{\Delta(t)}}{X(t)} = \frac{\sqrt{\Delta(-1/y)}}{X(-1/y)} > \frac{y^2 (1-\varepsilon)^{1/2}}{1-y^2}.
\]
Also we can show that
\[
\frac{Z(t)}{\Delta(t)} < 5 n^{3/2} y^{2n-4} (1-y^2)^3.
\]
Hence as in (1.36) it is easy to conclude that
\[
\text{EN}_n(-\infty,-1) \geq \frac{1}{2\pi} \log n + O \left( \frac{K^2}{n^2} \right)
\]
In order to estimate an upper bound for \(\text{EN}_n(-\infty,-1)\), we apply procedure described in 1.3 after putting \(y=-1/t\).

Thus
\[
\int_{0}^{1} I_2(y) \frac{dy}{y^2} < \frac{n^{1/2} |K|}{\sqrt{2\pi (n - 2)}} \exp \left(-\sqrt{n}\right),
\]
and
\[
\int_{1}^{n} I_2(y) \frac{dy}{y^2} < \frac{n^{1/2} |K|}{\sqrt{2\pi (n - 2)}} \left\{ n \left( n - \frac{1}{\sqrt{n}} \right) \right\}^{-1/2}.
\]
So
\[
\int_{-\infty}^{1} I_2(t) dt = o(1).
\] (1.44)

Like (1.13), we have
\[
\int_{-\infty}^{1} I_2(t) dt \leq \frac{1}{\pi} \int_{-\infty}^{1} \frac{\sqrt{\Delta(y)}}{y^2} dy
\]
\[
< \frac{1}{2\pi} \log n + O(1).
\] (1.45)

From (1.44) and (1.45), we have
\[
\text{EN}_n(-\infty, -1) \sim \frac{1}{2\pi} \log n 
\] (1.46)

This completes the proof of the theorem.

References


